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Explicit Quadrature Formulae for Entire Functions of Exponential Type

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We obtain, for entire functions of exponential type, a complementary result and a generalization of a quadrature formula with nodes at the zeros of Bessel functions. Our formula contains a sequence of rational fractions whose properties are studied. © 1998 Academic Press

1. INTRODUCTION

Given any complex number α, the function

$$\frac{J_\alpha(z)}{z^\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{\alpha+2k} k!\, \Gamma(k+\alpha+1)} z^{2k} \tag{1}$$

is an even entire function of exponential type 1. Here $J_\alpha(z)$ is the Bessel function of the first kind of order α. Let $j_k = j_k(\alpha)$, $k = \pm 1, \pm 2, \ldots$, be the zeros of $J_\alpha(z)/z^\alpha$ ordered such that $j_{-k} = -j_k$ and $0 < |j_1| \leqslant |j_2| \leqslant \cdots$.

An exact quadrature formula with zeros of Bessel functions as nodes has been proved in [2].

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THEOREM A. *Let $\Re(\alpha) > -1$. For all entire functions f of exponential type 2τ such that $f(x) = O(|x|^{-\delta})$, $x \to \pm\infty$, with $\delta > 2\Re(\alpha) + 2$, we have*

$$\int_0^\infty x^{2\alpha+1}(f(x) + f(-x))\,dx$$
$$= \frac{2}{\tau^{2\alpha+2}} \sum_{k=1}^{\infty} \frac{j_k^{2\alpha}}{(J'_\alpha(j_k))^2} \left(f\left(\frac{j_k}{\tau}\right) + f\left(-\frac{j_k}{\tau}\right)\right). \tag{2}$$

The growth condition on f has been relaxed in [3] assuming that $\alpha > -1$.

THEOREM B. *If $\alpha > -1$ then (2) holds for every entire function f of exponential type 2τ such that $x^{2\alpha+1}(f(x) + f(-x))$ belongs to $L^1[0, \infty)$.*

In this paper we first obtain a complementary result related to (2). We also give a result which may be seen as a generalization of (2).

2. STATEMENT OF THE RESULTS

We note that the right-hand side of (2) vanishes whenever $f(z)$ is replaced by $(J_\alpha(\tau z)/(\tau z)^\alpha)\, f(z)$. Also, the asymptotic formula [6, Sect. 7.21]

$$J_\alpha(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{|z|^{3/2}}\right), \qquad z \in \mathbb{R}, \qquad z \to \infty$$

implies that $J_\alpha(x) = O(|x|^{-1/2})$, $x \to \pm\infty$. Thus, if f is an entire function of exponential type τ such that $f(x) = O(|x|^{-\delta})$, $x \to \pm\infty$, with $\delta > \Re(\alpha) + \frac{3}{2}$, $\Re(\alpha) > -1$, then

$$\int_0^\infty x^{\alpha+1} J_\alpha(\tau x)(f(x) + f(-x))\,dx = 0. \tag{3}$$

Applying (3) to a function of the form $1/x^2(f(x) - f(0))$, where f may be supposed to be even, and using the formula (see [6, p. 391]) $\int_0^\infty x^{\alpha-1} J_\alpha(x)\,dx = 2^{\alpha-1}\Gamma(\alpha)$, we obtain

$$f(0) = \frac{\tau^\alpha}{2^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} J_\alpha(\tau x)(f(x) + f(-x))\,dx. \tag{4}$$

The particular case $\alpha = -\frac{1}{2}$ of (3), applied to $f(z)$ and $f(z-(\pi/2\tau))$, readily leads us to the formula [4, p. 109]

$$\int_{-\infty}^{\infty} e^{i\tau x} f(x)\, dx = 0.$$

Also, applying (4) with $\alpha = \frac{1}{2}$, we obtain the well-known formula [5, p. 109]

$$f(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(\tau x)}{x} f(x)\, dx.$$

Note that we obtain the last two formulae under the growth condition $f(x) = O(|x|^{-\delta})$, $x \to \pm\infty$, respectively with $\delta > 1$ and $\delta > 0$, although these formulae are valid for more general functions.

Our first theorem is a generalization of (4).

THEOREM 1. *Let p be a nonnegative integer and $\Re(\alpha) > p$. For all entire functions f of exponential type τ such that $f(x) = O(|x|^{-\delta})$, $x \to \pm\infty$, with $\delta > \Re(\alpha) - 2p - \frac{1}{2}$, we have*

$$\int_{0}^{\infty} x^{\alpha - 2p - 1} J_\alpha(\tau x)(f(x) + f(-x))\, dx$$
$$= \frac{1}{\tau^{\alpha - 2p}} \sum_{j=0}^{p} \frac{2^{\alpha - 2p + 2j} \Gamma(\alpha - p + j)}{(p-j)!} \frac{f^{(2j)}(0)}{\tau^{2j}(2j)!}. \tag{5}$$

The function defined by (1) is called Spherical Bessel function whenever $\alpha = n + \frac{1}{2}$, n being an integer. Since $\Gamma(n+\frac{1}{2}) = \sqrt{\pi}\,(2n)!/2^{2n} n!$, $n = 0, 1, 2, ...$, we deduce from Theorem 1 the following

COROLLARY. *Let p and n be nonnegative integers, $p \leqslant n$. For all functions f of exponential type τ such that $f(x) = O(|x|^{-\delta})$, $x \to \pm\infty$, with $\delta > n - 2p$, we have*

$$\int_{-\infty}^{\infty} x^{n - 2p - 1/2} J_{n+1/2}(\tau x)\, f(x)\, dx$$
$$= \frac{\sqrt{2\pi}}{\tau^{n - 2p + 1/2} 2^n n!} \sum_{j=0}^{p} \binom{n}{p-j} (2n - 2p + 2j)! \frac{f^{(2j)}(0)}{\tau^{2j}(2j)!}. \tag{6}$$

For $p = n$, Eq. (6) reduces (with $\delta > -n$) to

$$\int_{-\infty}^{\infty} \frac{J_{n+1/2}(\tau x)}{x^{n+1/2}} f(x)\, dx = \frac{\sqrt{2\pi}\, \tau^{n-1/2}}{2^n n!} \sum_{j=0}^{p} \binom{n}{j} \frac{f^{(2j)}(0)}{\tau^{2j}}. \tag{7}$$

Before we state our second theorem, it is convenient to introduce some notations. We set $N(0) := 0$, $N(1) := 0$ and

$$N(p) := \sum_{v=2}^{p} [p/v] \qquad \text{for} \quad p \geqslant 2. \tag{8}$$

Here, $[a]$ is the integral part of the real number a. We also set $u(0; \alpha) := 1/\alpha$ and

$$u(p; \alpha) := \left(2^{[3p/2]} p! \, (\alpha - p) \prod_{v=1}^{p} (\alpha + v)^{[p/v]} \right)^{-1} \qquad \text{for} \quad p \geqslant 1. \tag{9}$$

THEOREM 2. *Let p be a nonnegative integer and $\Re(\alpha) > p$. For all entire functions f of exponential type 2τ such that $f(x) = O(|x|^{-\delta})$, $x \to \pm\infty$, with $\delta > 2\Re(\alpha) - 2p$, we have*

$$\begin{aligned}
&\int_0^\infty x^{2\alpha - 2p - 1}(f(x) + f(-x)) \, dx \\
&\quad = \frac{2}{\tau^{2\alpha - 2p}} \sum_{k=1}^{\infty} \frac{j_k^{2\alpha - 2p - 2}}{(J'_\alpha(j_k))^2} \left(f\left(\frac{j_k}{\tau}\right) + f\left(-\frac{j_k}{\tau}\right) \right) \\
&\qquad + \frac{2^{2\alpha}(\Gamma(\alpha + 1))^2}{\tau^{2\alpha - 2p}} \sum_{j=0}^{p} u(p - j; \alpha) \, R_{N(p-j)}(\alpha) \frac{f^{(2j)}(0)}{\tau^{2j}(2j)!},
\end{aligned} \tag{10}$$

where $R_{N(j)}(\alpha)$ is a polynomial in α of degree $N(j)$ for $0 \leqslant j \leqslant p$, whose leading coefficient is $2^{[j/2]}$.

Equation (10), applied to a function of the form $(J_\alpha(\tau z)/(\tau z)^\alpha) \, f(z)$, gives a non explicit version of (5). In fact, Theorem 1 will be used to obtain informations on the sequence $e(p; \alpha)$, $p = 0, 1, 2, ...,$ defined by

$$e(p; \alpha) := u(p, \alpha) \, R_{N(p)}(\alpha). \tag{11}$$

The sequence $e(p; \alpha)$, $p = 0, 1, 2, ...,$ satisfies the following recurrence relation.

THEOREM 3. *Let α be a complex number, $\alpha \neq 0, \pm 1, \pm 2, ...,$ and let p be a nonnegative integer. We have $e(0; \alpha) = 1/\alpha$ and*

$$\begin{aligned}
e(p + 1; \alpha) &= \sum_{k=0}^{p} \frac{(-1)^k \, \Gamma(\alpha + 1) \, e(p - k; \alpha)}{2^{2k+2}(k + 1)! \, \Gamma(\alpha + k + 2)} \\
&\quad + \frac{\Gamma(\alpha - p - 1)}{2^{2p+2}(p + 1)! \, \Gamma(\alpha + 1)}.
\end{aligned} \tag{12}$$

From Theorem 3 we will deduce the generating function of the sequence $e(p; \alpha)$, $p = 0, 1, 2, \ldots$. This generating function will be used to prove that the $R_{N(p)}(\alpha)$ of (10) are polynomials in α.

THEOREM 4. *Let* $\phi_\alpha(z) := 2^\alpha \Gamma(\alpha+1)(J_\alpha(z)/z^\alpha)$. *We have, for* $\alpha \neq 0, \pm 1, \pm 2, \ldots$,

$$\frac{\phi_{-\alpha}(\sqrt{z})}{\alpha\phi_\alpha(\sqrt{z})} = \sum_{p=0}^{\infty} e(p; \alpha)\, z^p, \tag{13}$$

where the series converges for $|z| < |j_1(\alpha)|^2$.

3. LEMMAS

The proof of Theorem 1 is based on the following result.

LEMMA 1. *Let* p *be a nonnegative integer and* $p < \Re(\alpha) < 2p + \frac{1}{2}$. *If* $-1 \leqslant \lambda \leqslant 1$ *then we have*

$$\int_0^\infty x^{\alpha-2p-1} J_\alpha(x) \cos(\lambda x)\, dx = \sum_{j=0}^{p} \frac{(-1)^j\, 2^{\alpha-2p+2j-1} \Gamma(\alpha-p+j)\, \lambda^{2j}}{(p-j)!\,(2j)!}. \tag{14}$$

Proof. Let β, μ, ν be complex numbers and let a, b be real numbers. It is known [6, Sect. 13.4] that

$$\int_0^\infty \frac{J_\mu(ax)\, J_\nu(bx)}{x^\beta}\, dx = \frac{2^{-\beta} b^\nu \Gamma(1/2(\mu+\nu-\beta+1))}{a^{\nu-\beta+1}\Gamma(\nu+1)\,\Gamma(1/2(\beta+\mu-\nu+1))}$$
$$\times F\left(\frac{1}{2}(\mu+\nu-\beta+1), \frac{1}{2}(\nu-\beta-\mu+1); \nu+1; \frac{b^2}{a^2}\right) \tag{15}$$

provided that $0 < b < a$ and that the integral converges. Here,

$$F(\alpha, \beta; \gamma; z) := 1 + \frac{\alpha\beta}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \cdots$$

is the Gaussian hypergeometric function. We take, in (15), $\mu = \alpha$, $\nu = -\frac{1}{2}$, $a = 1$, $b = \lambda$ and $\beta = -\alpha + 2p + \frac{1}{2}$. We obtain

$$\int_0^\infty x^{\alpha-2p-1} J_\alpha(x) \cos(\lambda x)\, dx = \frac{2^{\alpha-2p-1}}{p!}\, \Gamma(\alpha-p)\, F\left(\alpha-p, -p; \frac{1}{2}; \lambda^2\right),$$

from which (14) follows for $0<\lambda<1$. The result holds by continuity for $\lambda=0, 1$ and by symmetry for $-1 \leqslant \lambda \leqslant 0$. ∎

The basic idea, in the proof of Theorem 2, is to apply (2) repeatedly to a function of the form

$$g_\alpha(z) := \frac{1}{z^2}\left(f(z) - \left(2^\alpha \Gamma(\alpha+1)\,\frac{J_\alpha(z)}{z^\alpha}\right)^2 f(0)\right), \tag{16}$$

where $f(z)$ is an even entire function.

LEMMA 2. *Let f be an even entire function and j a nonnegative integer. We have*

$$g_\alpha^{(2j)}(0) = \frac{1}{(2j+2)(2j+1)}\, f^{(2j+2)}(0) + \frac{(-1)^j\,(2j)!\,(\Gamma(\alpha+1))^2\,\Gamma(2\alpha+2j+3)}{2^{2j+2}(j+1)!\,\Gamma(2\alpha+j+2)(\Gamma(\alpha+j+2))^2}\, f(0). \tag{17}$$

Proof. We use the formula [6, Sect. 5.4]

$$\frac{J_\alpha^2(z)}{z^{2\alpha}} = \sum_{j=0}^{\infty} \frac{(-1)^j\,\Gamma(2\alpha+2j+1)}{2^{2\alpha+2j} j!\,\Gamma(2\alpha+j+1)(\Gamma(\alpha+j+1))^2}\, z^{2j}$$

in conjunction with

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(2j)}(0)}{(2j)!}\, z^{2j}. \quad ∎$$

We wil also need the following result [1, p. 148], known as Faa Di Bruno's formula.

LEMMA 3. *We have, for $j = 1, 2, ...,$*

$$(F(G(z)))^{(j)} = \sum_{r=1}^{j} \sum_{\pi(j,r)} c(k_1, ..., k_j)\, F^{(r)}(G(z)) \prod_{\nu=1}^{j} (G^{(\nu)}(z))^{k_\nu}, \tag{18}$$

where $c(k_1, ..., k_j) := j!/k_1! \cdots k_j!\,(1!)^{k_1} \cdots (j!)^{k_j}$ and $\pi(j, r)$ means that the summation is extended over all nonnegative integers $k_1, ..., k_j$ such that $k_1 + 2k_2 + \cdots + jk_j = j$ and $k_1 + k_2 + \cdots + k_j = r$.

Finally, we prove a crucial lemma, the first step in the proof of Theorem 2.

LEMMA 4. *Under the hypothesis of Theorem* 2, *we have*

$$\int_0^\infty x^{2\alpha-2p-1}(f(x)+f(-x))\,dx$$

$$=\frac{2}{\tau^{2\alpha-2p}}\sum_{k=1}^{\infty}\frac{j_k^{2\alpha-2p-2}}{(J'_\alpha(j_k))^2}\left(f\left(\frac{j_k}{\tau}\right)+f\left(-\frac{j_k}{\tau}\right)\right)$$

$$+\frac{2^{2\alpha}(\Gamma(\alpha+1))^2}{\tau^{2\alpha-2p}}\sum_{j=0}^{p}e(p-j;\alpha)\,\frac{f^{(2j)}(0)}{\tau^{2j}(2j)!},\tag{19}$$

where the $e(j;\alpha)$, $0\leqslant j\leqslant p$, *are rational fractions in* α.

Proof. Without loss of generality, we may suppose that $f(z)$ is even and $\tau=1$. The function $g_\alpha(z)$ defined by (16) is then an even entire function of exponential type 2.

Now we prove the lemma by induction on p. For $p=0$, we use (2) where f is replaced by g_α; we obtain

$$\int_0^\infty x^{2\alpha+1}g_\alpha(x)\,dx=2\sum_{k=1}^{\infty}\frac{j_k^{2\alpha-2}}{(J'_\alpha(j_k))^2}\,f(j_k).\tag{20}$$

For $\Re(\alpha)>0$ this equality may be written in the form

$$\int_0^\infty x^{2\alpha-1}f(x)\,dx=2\sum_{k=1}^{\infty}\frac{j_k^{2\alpha-2p-4}}{(J'_\alpha(j_k))^2}\,f(j_k)+\frac{2^{2\alpha}(\Gamma(\alpha+1))^2}{2\alpha}\,f(0)\tag{21}$$

since [6, p. 403]

$$\int_0^\infty \frac{J_\alpha^2(x)}{x^{2k+1}}\,dx=\frac{(2k)!\;\Gamma(\alpha-k)}{2^{2k+1}(k!)^2\;\Gamma(\alpha+k+1)}\tag{22}$$

for $k=0, 1, 2, \ldots$ and $\Re(\alpha)>k$. Thus the lemma is valid for $p=0$ and $e(0;\alpha)=1/\alpha$.

Suppose that (19) holds for some positive integer p. Replacing f by g_α in (19), we obtain

$$2\int_0^\infty x^{2\alpha-2p-1}g_\alpha(x)\,dx$$

$$=4\sum_{k=1}^{\infty}\frac{j_k^{2\alpha-2p-4}}{(J'_\alpha(j_k))^2}\,f(j_k)+2^{2\alpha}(\Gamma(\alpha+1))^2\sum_{j=0}^{p}e(p-j;\alpha)\,\frac{g_\alpha^{(2j)}(0)}{(2j)!}.\tag{23}$$

For $\Re(\alpha) > p+1$ this equality may be written, by Lemma 2 and (22), in the form

$$2\int_0^\infty x^{2\alpha-2p-3} f(x)\,dx$$
$$= 4\sum_{k=1}^\infty \frac{j_k^{2\alpha-2p-4}}{(J'_\alpha(j_k))^2} f(j_k)$$
$$+ 2^{2\alpha}(\Gamma(\alpha+1))^2 \left(\sum_{j=0}^p e(p-j;\alpha) \frac{f^{(2j+2)}(0)}{(2j+2)!} + e(p+1;\alpha)\, f(0) \right), \tag{24}$$

where

$$e(p+1;\alpha) := \sum_{j=0}^p \frac{(-1)^j (\Gamma(\alpha+1))^2\, \Gamma(2\alpha+2j+3)\, e(p-j;\alpha)}{2^{2j+2}(j+1)!\, \Gamma(2\alpha+j+2)(\Gamma(\alpha+j+2))^2}$$
$$+ \frac{(2p+2)!\, \Gamma(\alpha-p-1)}{2^{2p+2}((p+1)!)^2\, \Gamma(\alpha+p+2)}. \tag{25}$$

Obviously, if each $e(p-j;\alpha)$, $0 \leqslant j \leqslant p$, is a rational fraction in α, so is $e(p+1;\alpha)$. This completes the proof of the lemma. ∎

4. PROOFS OF THE THEOREMS

Proof of Theorem 1. According to the classical theorem of Paley and Wiener, an entire function of exponential type τ, which belongs to $L^2(-\infty, \infty)$, has a representation of the form

$$f(z) = \int_{-\tau}^{\tau} e^{izt} \psi(t)\, dt, \tag{26}$$

where $\psi \in L^2(-\tau, \tau)$. So, using Lemma 1,

$$\int_0^\infty x^{\alpha-2p-1} J_\alpha(x) \left(f\left(\frac{x}{\tau}\right) + f\left(-\frac{x}{\tau}\right) \right) dx$$
$$= 2\int_0^\infty \int_{-\tau}^{\tau} x^{\alpha-2p-1} J_\alpha(x) \cos\left(\frac{tx}{\tau}\right) \psi(t)\, dt\, dx$$
$$= \int_{-\tau}^{\tau} \sum_{j=0}^p \frac{(-1)^j\, 2^{\alpha-2p+2j} \Gamma(\alpha-p+j)}{(p-j)!\, (2j)!\, \tau^{2j}} t^{2j} \psi(t)\, dt$$
$$= \sum_{j=0}^p \frac{2^{\alpha-2p+2j} \Gamma(\alpha-p+j)}{(p-j)!} \frac{f^{(2j)}(0)}{\tau^{2j}(2j)!}, \tag{27}$$

which proves Theorem 1 in the case $f \in L^2(-\infty, \infty)$. In general we consider a function of the form $(\sin(\varepsilon z)/\varepsilon z)^m f(z)$, for some positive integer m, and let $\varepsilon \to 0$. The passage to the limit is easily justifiable. ▮

The relation (25) is a recurrence relation for the sequence $e(p; \alpha)$, $p = 0, 1, \ldots$. Theorem 3 gives a simpler one.

Proof of Theorem 3. Given an entire function of exponential type $\tau = 1$ such that $f(x) = O(|x|^{-\delta})$, $x \to \pm\infty$, with $\delta > \Re(\alpha) - 2p - 1/2$, the function $h(z) := (J_\alpha(z)/z^\alpha) f(z)$ satisfies the hypothesis of Lemma 4. Hence,

$$\begin{aligned}
&\int_0^\infty x^{\alpha-2p-1} J_\alpha(x)(f(x)+f(-x))\,dx \\
&\quad = 2^{2\alpha}(\Gamma(\alpha+1))^2 \sum_{j=0}^{p} e(p-j;\alpha)\,\frac{h^{(2j)}(0)}{(2j)!} \\
&\quad = 2^{\alpha}(\Gamma(\alpha+1))^2 \sum_{j=0}^{p}\sum_{k=0}^{j} \frac{(-1)^{j-k}\, e(p-j;\alpha)}{2^{2j-2k}(j-k)!\,\Gamma(\alpha+j-k+1)}\,\frac{f^{(2j)}(0)}{(2j)!} \\
&\quad = 2^{\alpha}(\Gamma(\alpha+1))^2 \sum_{j=0}^{p}\sum_{k=0}^{p-j} \frac{(-1)^{k}\, e(p-j-k;\alpha)}{2^{2k}k!\,\Gamma(\alpha+k+1)}\,\frac{f^{(2j)}(0)}{(2j)!}.
\end{aligned} \tag{28}$$

It follows from Theorem 1 that

$$\begin{aligned}
&(\Gamma(\alpha+1))^2 \sum_{j=0}^{p}\sum_{k=0}^{p-j} \frac{(-1)^{k}\, e(p-j-k;\alpha)}{2^{2k}k!\,\Gamma(\alpha+k+1)}\,\frac{f^{(2j)}(0)}{(2j)!} \\
&\quad = \sum_{j=0}^{p} \frac{2^{2j-2p}\Gamma(\alpha-p+j)}{(p-j)!}\,\frac{f^{(2j)}(0)}{(2j)!}.
\end{aligned} \tag{29}$$

Since f is arbitrary, we conclude that

$$(\Gamma(\alpha+1))^2 \sum_{k=0}^{p-j} \frac{(-1)^{k}\, e(p-j-k;\alpha)}{2^{2k}k!\,\Gamma(\alpha+k+1)} = \frac{2^{2j-2p}\Gamma(\alpha-p+j)}{(p-j)!}, \qquad 0 \leqslant j \leqslant p, \tag{30}$$

which is equivalent to (12) for $\Re(\alpha) > p$. Since both sides of (12) are rational fractions in α, it is clear that this formula remains valid for any $\alpha \neq 0, \pm 1, \pm 2, \ldots$. ▮

Proof of Theorem 4. Let $E_\alpha(z) := \sum_{p=0}^{\infty} e(p; \alpha)\, z^p$. Multiplying both sides of (12) by z^{p+1} and summing over $p = 0, 1, 2, \ldots$, we obtain

$$\sum_{p=0}^{\infty} e(p+1;\alpha)\, z^{p+1} = \sum_{p=0}^{\infty} \sum_{k=0}^{p} \frac{(-1)^k\, \Gamma(\alpha+1)\, e(p-k;\alpha)}{2^{2k+2}(k+1)!\, \Gamma(\alpha+k+2)}\, z^{p+1} + \sum_{p=0}^{\infty} \frac{\Gamma(\alpha-p-1)}{2^{2p+2}(p+1)!\, \Gamma(\alpha+1)}\, z^{p+1}. \tag{31}$$

Permutting the order of summation, we get, after some obvious changes of variables,

$$E_\alpha(z) - \frac{1}{\alpha} = E_\alpha(z) \sum_{p=1}^{\infty} \frac{(-1)^{p-1}\, \Gamma(\alpha+1)}{2^{2p} p!\, \Gamma(\alpha+p+1)}\, z^p + \sum_{p=1}^{\infty} \frac{\Gamma(\alpha-p)}{2^{2p} p!\, \Gamma(\alpha+1)}\, z^p, \tag{32}$$

whence

$$E_\alpha(z) \sum_{p=0}^{\infty} \frac{(-1)^p\, \Gamma(\alpha+1)}{2^{2p} p!\, \Gamma(\alpha+p+1)}\, z^p = \sum_{p=0}^{\infty} \frac{\Gamma(\alpha-p)}{2^{2p} p!\, \Gamma(\alpha+1)}\, z^p. \tag{33}$$

We have

$$\sum_{p=0}^{\infty} \frac{(-1)^p}{2^{2p} p!\, \Gamma(\alpha+p+1)}\, z^p = 2^\alpha \frac{J_\alpha(\sqrt{z})}{(\sqrt{z})^\alpha}.$$

Also, using the relation $\Gamma(w)\,\Gamma(1-w) = \pi/\sin(\pi w)$ with $w = \alpha - p$, we see that

$$\sum_{p=0}^{\infty} \frac{\Gamma(\alpha-p)}{2^{2p} p!}\, z^p = \frac{\pi 2^{-\alpha}}{\sin(\pi\alpha)} \frac{J_{-\alpha}(\sqrt{z})}{(\sqrt{z})^{-\alpha}}.$$

We then deduce from (33) that

$$E_\alpha(z) = \frac{\pi}{2^{2\alpha} \sin(\pi\alpha)(\Gamma(\alpha+1))^2} \frac{z^\alpha J_{-\alpha}(\sqrt{z})}{J_\alpha(\sqrt{z})}, \tag{34}$$

which reduces to (13) after another application of the relation $\Gamma(w)\,\Gamma(1-w) = \pi/\sin(\pi w)$ with $w = -\alpha$. ∎

Proof of Theorem 2. It remains to prove, in view of Lemma 4, that the functions $R_{N(p)}(\alpha)$, $p = 0, 1, 2, \ldots$, appearing in the right-hand side of (10), are polynomials in α of degree $N(p)$ with leading coefficient $2^{[p/2]}$. The function $R_{N(p)}(\alpha)$ is related to $e(p;\alpha)$ by (11); it is thus a rational fraction in α.

Our first goal is to obtain an explicit formula for $R_{N(p)}(\alpha)$. It is clear, from Theorem 3, that $R_{N(0)}(\alpha) = 1$. We may thus assume that p is a positive integer. We have, using the generating function (13),

$$p!\, e(p;\alpha)=\frac{1}{\alpha}\left(\frac{\phi_{-\alpha}(\sqrt{z})}{\phi_{\alpha}(\sqrt{z})}\right)^{(p)} \qquad (z=0)$$

$$=\frac{\Gamma(\alpha-p)}{2^{2p}\Gamma(\alpha+1)}+\sum_{j=1}^{p}\binom{p}{j}\frac{\Gamma(\alpha-p+j)}{2^{2p-2j}\Gamma(\alpha+1)}\left(\frac{1}{\phi_{\alpha}(\sqrt{z})}\right)^{(j)}$$

$$(z=0). \tag{35}$$

We use Lemma 3 with $F(z)=1/z$, $G(z)=\phi_{\alpha}(\sqrt{z})$ and $z=0$. We obtain a formula for $(1/\phi_{\alpha}(\sqrt{z}))^{(j)}$ $(z=0)$ which, once we substitute in (35), results in

$$p!\, e(p;\alpha)=\frac{\Gamma(\alpha-p)}{2^{2p}\Gamma(\alpha+1)}+\sum_{j=1}^{p}\sum_{r=1}^{j}\sum_{\pi(j,r)}\binom{p}{j}(-1)^{r+j}r!\, c(k_1,\ldots,k_j)$$
$$\times\frac{\Gamma(\alpha-p+j)}{2^{2p}\Gamma(\alpha+1)}\prod_{v=1}^{j}\left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+v+1)}\right)^{k_v}. \tag{36}$$

So, we obtain the explicit formula

$$R_{N(p)}(\alpha)=2^{[3p/2]}(\alpha-p)\prod_{v=1}^{p}(\alpha+v)^{[p/v]}\left(\frac{\Gamma(\alpha-p)}{2^{2p}\Gamma(\alpha+1)}+\sum_{j=1}^{p}\sum_{r=1}^{j}\sum_{\pi(j,r)}\binom{p}{j}\right.$$
$$\left.\times(-1)^{r+j}r!\, c(k_1,\ldots,k_j)\frac{\Gamma(\alpha-p+j)}{2^{2p}\Gamma(\alpha+1)}\prod_{v=1}^{j}\left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+v+1)}\right)^{k_v}\right). \tag{37}$$

In (37), each term of the form

$$\prod_{v=1}^{p}(\alpha+v)^{[p/v]}\prod_{v=1}^{j}\left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+v+1)}\right)^{k_v}=\prod_{v=1}^{p}(\alpha+v)^{[p/v]}\prod_{v=1}^{j}(\alpha+v)^{-(k_v+\cdots+k_j)}$$

is, for $1\leqslant j\leqslant p$, a polynomial in α; indeed, we have

$$k_v+\cdots+k_j\leqslant\frac{1}{v}(vk_v+\cdots+jk_j)\leqslant j/v\leqslant p/v, \qquad 1\leqslant v\leqslant j,$$

and so $[p/v]-(k_v+\cdots+k_j)\geqslant 0$. It then follows from (37) that the only possible poles of the rational fraction $R_{N(p)}(\alpha)$ are $\alpha=0, 1, \ldots, (p-1)$.

Now we will show that

$$\lim_{\alpha\to k}(\alpha-k)\, R_{N(p)}(\alpha)=0, \qquad k=0, 1, \ldots, (p-1), \tag{38}$$

from which we infer that no pole of $R_{N(p)}(\alpha)$ could exist.

The left-hand side of (38) can be evaluated from (37); we find that

$$\lim_{\alpha \to k} (\alpha - k)\, R_{N(p)}(\alpha) = \frac{(-1)^{p-k}\, 2^{[3p/2]-2p}(k-p)}{k!} \prod_{v=1}^{p} (k+v)^{[p/v]}$$
$$\times \left(\frac{1}{(p-k)!} + \sum_{j=1}^{p-k} \sum_{r=1}^{j} \sum_{\pi(j,r)} \binom{p}{j} \right.$$
$$\left. \times \frac{(-1)^r\, r!\, c(k_1, ..., k_j)}{(p-k-j)!} \prod_{v=1}^{j} \left(\frac{k!}{(k+v)!} \right)^{k_v} \right). \qquad (39)$$

Next we will prove that the right-hand side of (39) is equal to zero. Let

$$h_k(z) := \frac{J_k(2i\sqrt{z})}{(2i\sqrt{z})^k} = \sum_{v=0}^{\infty} \frac{1}{2^k v!\, (k+v)!}\, z^v.$$

We have, for $k = 0, 1, ..., p-1$,

$$0 = (z^k)^{(p)}\, (z=0) = \left(\frac{z^k h_k(z)}{h_k(z)} \right)^{(p)} (z=0)$$
$$= \sum_{j=0}^{p} \binom{p}{j} (z^k h_k(z))^{(p-j)}\, (z=0) \left(\frac{1}{h_k(z)} \right)^{(j)} (z=0)$$
$$= k! \left(\frac{1}{(p-k)!} + \sum_{j=1}^{p-k} \sum_{r=1}^{j} \sum_{\pi(j,r)} \binom{p}{j} \frac{(-1)^r\, r!\, c(k_1, ..., k_j)}{(p-k-j)!} \prod_{v=1}^{j} \left(\frac{k!}{(k+v)!} \right)^{k_v} \right).$$

Thus (38) holds. It is clear from (37) that $R_{N(p)}(\alpha)$ has degree $N(p) = \sum_{v=2}^{p} [p/v]$. Its leading coefficient is

$$\lim_{\alpha \to \infty} \frac{R_{N(p)}(\alpha)}{\alpha^{N(p)}} = 2^{[3p/2]-2p} \left(1 + \sum_{j=1}^{p} \sum_{r=1}^{j} \sum_{\pi(j,r)} \binom{p}{j} (-1)^{r+j} r!\, c(k_1, ..., k_j) \right)$$
$$= 2^{[p/2]},$$

where the last step uses Lemma 3 with $F(z) = 1/z$, $G(z) = e^{-z}$ and $z = 0$. This completes the proof of Theorem 2. ∎

5. CONCLUDING REMARKS

In this final section we make some additional comments concerning our results and we give a few examples.

The special case $\alpha = 2p+1$ of Theorem 1 leads us to the following result: if f is an entire function of exponential type τ such that $f(x) = O(|x|^{-\delta})$, $x \to \pm\infty$, with $\delta > \frac{1}{2}$, then we have

$$\int_0^\infty J_{2p+1}(\tau x)(f(x)+f(-x))\,dx = \sum_{j=0}^{p} \binom{p+j}{2j} \left(\frac{2}{\tau}\right)^{2j+1} f^{(2j)}(0), \qquad p = 0, 1, ... \tag{40}$$

Applying (5) with $p = 0$ to a function of the form $\prod_{j=1}^{k} J_{\mu_j}(a_j z)/z^{\mu_j}$, we obtain the following known result [6, p. 419]:

$$\int_0^\infty x^{\alpha - M - 1} J_\alpha(\tau x) \prod_{j=1}^{k} J_{\mu_j}(a_j x)\,dx = 2^{\alpha - M - 1} \frac{\Gamma(\alpha)}{\tau^\alpha} \prod_{j=1}^{k} \frac{a_j^{\mu_j}}{\Gamma(\mu_j + 1)}, \tag{41}$$

where $M := \sum_{j=1}^{k} \mu_j$, $\sum_{j=1}^{k} |a_j| < \tau$ and $0 < \Re(\alpha) < \Re(M) + k/2 + 1/2$.

The first polynomials appearing in Theorem 2 are $R_0(\alpha) = 1$, $R_1(\alpha) = 2\alpha + 5$, $R_2(\alpha) = 2\alpha^2 + 13\alpha + 23$, $R_4(\alpha) = 4\alpha^4 + 56\alpha^3 + 303\alpha^2 + 748\alpha + 677$, $R_5(\alpha) = 4\alpha^5 + 84\alpha^4 + 731\alpha^3 + 3319\alpha^2 + 7821\alpha + 7313$. It is empirically evident that the coefficients of $R_{N(p)}(\alpha)$ are positive integers. Finally, we note that $e(p; \alpha) = O(1/\alpha^{p+1})$ as $\alpha \to \infty$.

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